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Fuel Slosh in Skewed Tanks

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Previous treatments of fuel slosh coupling with aircraft dynamics are extended to the case in which fuel tanks are skewed to the direction of flight, as in swept wing tanks. General and linearized equations of motion are developed and example calculations are made based on the Boeing 747-100 airplane. Slosh appears to be possible for swept wings only over a limited pitch attitude range, dependent on wing dihedral. For the example airplane at landing approach attitudes, coupling cannot happen, because partial inboard main wing tank fuel is fixed by gravity at the inboard tank ends. The gravity fixity is removed for an assumed lower dihedral angle. Then, slosh coupling with the Dutch roll mode of motion is small, but an unstable real root develops, similar to the spiral mode. Fuel slosh coupling to the Dutch roll mode is even smaller for centerline (unskewed) tanks. Tank locations forward of the aircraft's center of gravity are destabilizing.

Nomenclature

A, B, C	= matrices in perturbation equations
a	= effective tank length in slosh direction
c, s	= cosine, sine functions
d	= fuel depth
e_f	= perturbation fuel motion, parallel to $\boldsymbol{m}$
$\boldsymbol{F}$	= force applied to sloshing fuel
g	= acceleration of gravity
h	= tank height
I_x, I_y, I_z, I_{xz}	= moments and product of inertia, at equilibrium
$\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$	= unit vectors in airplane body axis system
L, M, N	= moments applied to the airframe by sloshing fuel
L_p, Y_v, etc.	= stability derivatives
$\boldsymbol{l}, \boldsymbol{m}, \boldsymbol{n}$	= unit vectors in fuel slosh coordinate system
l_0, m_0	= equilibrium coordinates of fuel center of gravity
m	= aircraft total mass, including equilibrium fuel
m_f	= effective sloshing fuel mass
P, Q, R	= aircraft body axis angular velocities
p, r	= perturbation rolling and yawing velocities
$\boldsymbol{Q}$	= torque applied to airframe by sloshing fuel
$\boldsymbol{r}_f$	= fuel position vector
U, V, W	= aircraft body axis linear velocities
$\boldsymbol{u}, \boldsymbol{x}$	= control and state vectors
V_{t0}, V	= flight velocity
$\boldsymbol{X}$	= force applied to airframe by sloshing fuel
X, Y, Z	= components of the $\boldsymbol{X}$ vector on body axes
β	= sideslip angle
Γ	= wing panel dihedral angle
δ_a, δ_r	= aileron, rudder angles
ζ	= slosh damping ratio
Θ, θ	= pitch angle and perturbation pitch angle
Λ	= wing panel sweep angle
$\boldsymbol{\Phi}, \phi$	= bank angle and perturbation bank angle
ψ	= perturbation yaw angle
$\boldsymbol{\Omega}$	= aircraft body axis angular velocity
ω	= slosh undamped natural frequency

Subscripts

m comp	= component in the $\boldsymbol{m}$ direction
0	= steady, straight, symmetric flight

Introduction

FUEL slosh occurs when fuel in a partially filled tank sloshes around inside the tank in response to vehicle accelerations. As the tank walls contain the sloshing fuel, transient forces are transmitted to the walls and the vehicle, by the fuel. Fuel slosh can be a problem in airplane stability and control if the fuel modes of motion couple significantly with the airplane's normal modes of motion. The fuel slosh problem is worth examining because modern jet airplanes tend to have high ratios of fuel to gross weight. In the past, significant fuel slosh coupling with the Dutch roll mode of motion has occurred on the Douglas A4 Skyhawk,[1] the Cessna T-37A, and the Boeing KC-135A airplanes.

A particularly simple analytical approach to the problem of fuel slosh coupling with the modes of airplane motion was made possible by Graham.[2] He used the velocity potential for liquid in a rectangular open-top tank given by Lamb.[3] Graham modeled the sloshing fuel as a simple pendulum plus a fixed mass below the pendulum. The pendulum angle from the vertical is taken as the average fuel surface angular displacement in its fundamental mode of motion. The fuel's general motion has higher harmonics of shorter wavelengths, all of which are neglected to define the equivalent pendulum.

Schy[4] set up the fuel slosh problem without using the Graham pendulum model, by assuming fuel is carried in spherical tanks. Although spherical tanks are never seen in airplanes, Schy's model is altogether equivalent to Graham's model for conventional rectangular or prismatic tanks. Schy's calculations show significant coupling into the Dutch roll mode of an airplane when the sloshing fuel mass is one-fourth the weight of the airplane.

Luskin and Lapin[5] used the Graham equivalent pendulum model to analyze fuel slosh coupling with the longitudinal short- and long-period modes of motion. There is a slight loss in short-period mode damping, but the long-period or phugoid mode will not couple measurably with fuel in a partially filled tank unless the fore and aft dimension of the tank is impossibly long.

Fuel slosh was a major concern for some large liquid-fueled boost or launch vehicles, such as NASA's Saturn V. Launch vehicle dynamic fuel slosh problems have included coupling with controlled pitch and yaw modes of motion as well as with elastic body bending modes.

The present study extends the earlier aircraft work to the case of fuel tanks that are skewed to the direction of flight, typical of the wing tanks on long-range jet airplanes. Additional terms appear in the equations of coupled airplane and fuel motion for the skewed case.

Fuel Slosh Model

Model tests at the Northrop Advanced Development Division using water in long, narrow tanks showed that the primary slosh mode is along the tank long axis, regardless of the direction of excitation. That is, tank acceleration components normal to the tank long axis produced relatively high-frequency water motions along the tank short axis, motions that died out very quickly. Thus, the model for fuel sloshing in long skewed tanks represents only motions

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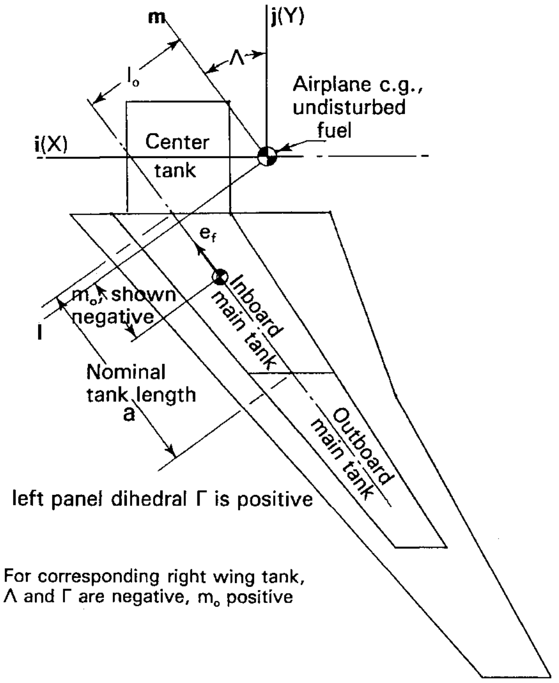

Fig. 1 Axes sets, unit vectors, and coordinates for skewed tanks (top view).

along the tank long axis, usually oriented to have the same sweep angles as the wing panels in which they are mounted.

Fuel Equation of Motion

Newton's second law of motion written for translation of a point mass with respect to translating, rotating axes provides the required mathematical model, as follows:

$$\ddot{\boldsymbol{r}}_f = \boldsymbol{F}/m_f - [\dot{\boldsymbol{V}} + \boldsymbol{\Omega}\times\boldsymbol{V} + \dot{\boldsymbol{\Omega}}\times\boldsymbol{r}_f + 2\boldsymbol{\Omega}\times\dot{\boldsymbol{r}}_f + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r}_f)] \tag{1}$$

Scalar fuel motion equations require Eq. (1) to be expressed in a common set of base vectors. Figure 1 shows the two axes sets in which the vectors are defined, the normal forward, right, and down X, Y, and Z aircraft body axis set with base vectors $\boldsymbol{i}, \boldsymbol{j}$, and $\boldsymbol{k}$, and the wing tank set defined by $\boldsymbol{l}, \boldsymbol{m}$, and $\boldsymbol{n}$. Both axes sets are centered at the aircraft center of gravity, with fuel at rest. The $\boldsymbol{l}, \boldsymbol{m}, \boldsymbol{n}$ set is rotated with respect to body axes by the wing tank sweep angle Λ. Fuel motion is assumed to take place in the $\boldsymbol{l}, \boldsymbol{m}$ plane, since vertical displacement, along the $\boldsymbol{n}$ axis, is negligible. Fuel slosh takes place parallel to the unit vector $\boldsymbol{m}$. The matrix transformation between the base vector sets is needed. This involves the large sweep angle Λ and the small dihedral angle Γ. Here, $c\Lambda$ and $s\Lambda$ are the sine and cosine of Λ:

$$\begin{bmatrix} \boldsymbol{l} \\ \boldsymbol{m} \\ \boldsymbol{n} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \Gamma \\ 0 & -\Gamma & 1 \end{bmatrix} \begin{bmatrix} c\Lambda & -s\Lambda & 0 \\ s\Lambda & c\Lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{i} \\ \boldsymbol{j} \\ \boldsymbol{k} \end{bmatrix} \tag{2}$$

Scalar Fuel Motion Equation

The angular and linear velocity vectors $\boldsymbol{\Omega}$ and $\boldsymbol{V}$ are expressed in the familiar body axis components P, Q, R, and U, V, W and require the transformation of Eq. (2) to obtain $\boldsymbol{l}, \boldsymbol{m}$, and $\boldsymbol{n}$ components. Also, from Fig. 1,

$$\boldsymbol{r}_f = l_0\boldsymbol{l} + (m_0 + e_f)\boldsymbol{m} \tag{3}$$

Fuel motions occur only parallel to the $\boldsymbol{m}$ axis, with positive displacement e_f parallel to positive $\boldsymbol{m}$. Thus, only the $\boldsymbol{m}$ component of the applied force vector $\boldsymbol{F}$ is of interest. This component has gravity, spring, and damper contributions, as follows:

$$(F/m_f)_{m\,\mathrm{comp}} = g(-s\Theta s\Lambda + s\Phi c\Theta c\Lambda + \Gamma c\Phi c\Theta) - 2\zeta\omega\dot{e}_f - \omega^2 e_f \tag{4}$$

The term on the left-hand side of Eq. (4) is the $\boldsymbol{m}$ component of $\boldsymbol{F}/m_f$. When the indicated cross products in Eq. (1) are performed and combined with Eq. (4), the resultant $\boldsymbol{m}$ component of $\boldsymbol{r}_f$ is the required fuel motion equation

$$\begin{aligned} \ddot{e}_f + 2\zeta\omega\dot{e}_f + \omega^2 e_f &= g(-s\Lambda s\Theta + s\Phi c\Theta c\Lambda + \Gamma c\Phi c\Theta) \\ &- \dot{U}s\Lambda - \dot{V}c\Lambda - \dot{W}\Gamma + (\Gamma P s\Lambda + \Gamma Q c\Lambda - R) \\ &\times (Uc\Lambda - Vs\Lambda) - (\Gamma U s\Lambda + \Gamma V c\Lambda - W)(Pc\Lambda - Qs\Lambda) \\ &+ l_0(\Gamma\dot{P}s\Lambda + \Gamma\dot{Q}c\Lambda - \dot{R}) \\ &- (m_0 + e_f)(\Gamma P s\Lambda + \Gamma Q c\Lambda - R)^2 \\ &- (m_0 + e_f)(Pc\Lambda - Qs\Lambda)^2 \\ &+ l_0(Pc\Lambda - Qs\Lambda)(Ps\Lambda + Qc\Lambda + \Gamma R) \end{aligned} \tag{5}$$

When Eq. (5) is applied to a right-wing fuel tank in a wing with positive dihedral, the angles Λ and Γ are negative, and m_0 is positive. For a left-wing panel (Fig. 1) the opposite signs apply. The sign of Γ is reversed for wings with negative geometric dihedral, or anhedral. All three parameters Λ, Γ, and m_0 are zero for the case of lateral or antisymmetric slosh in a centerline fuel tank.

Forces and Moments Transmitted to the Airframe

Using the assumed mass-spring-dashpot model of sloshing fuel, incremental forces applied to the airframe must come from the spring and dashpot connection to the airframe, or

$$d\boldsymbol{X}/m_f = \left(\omega^2 e_f + 2\zeta\omega\dot{e}_f\right)\boldsymbol{m} \tag{6}$$

Applied incremental moments resulting from sloshing fuel are

$$d\boldsymbol{Q} = \boldsymbol{r}_0 \times d\boldsymbol{X} + d\boldsymbol{r}_f \times \boldsymbol{X}_0 \tag{7}$$

The second term on the right-hand side of Eq. (7) accounts for the weight moment resulting from fuel mass shift $d\boldsymbol{r}_f$. The inverse of the transformation equations (2) gives the applied forces as

$$\begin{aligned} dX/m_f &= dX_{m\,\mathrm{comp}} s\Lambda/m_f = \left(\omega^2 e_f + 2\zeta\omega\dot{e}_f\right)s\Lambda \\ dY/m_f &= dX_{m\,\mathrm{comp}} c\Lambda/m_f = \left(\omega^2 e_f + 2\zeta\omega\dot{e}_f\right)c\Lambda \\ dZ/m_f &= dX_{m\,\mathrm{comp}} \Gamma/m_f = \left(\omega^2 e_f + 2\zeta\omega\dot{e}_f\right)\Gamma \end{aligned} \tag{8}$$

In Eq. (8), dX, dY, and dZ are the slosh forces applied to the airframe, on body axes. Here $dX_{m\,\mathrm{comp}}$ is the $\boldsymbol{m}$ component of these forces. The steady-state gravity force vector is

$$\boldsymbol{X}_0/m_f = -gs\Theta_0\boldsymbol{i} + gc\Theta_0\boldsymbol{k} \tag{9}$$

The vectors $\boldsymbol{r}_{f0} = l_0\boldsymbol{l} + m_0\boldsymbol{m}$ and $d\boldsymbol{r}_f = e_f\boldsymbol{m}$ are transformed to body axes by the inverse of Eq. (2) to give

$$\boldsymbol{r}_{f0} = (l_0 c\Lambda + m_0 s\Lambda)\boldsymbol{i} + (-l_0 s\Lambda + m_0 c\Lambda)\boldsymbol{j} + m_0\Gamma\boldsymbol{k} \tag{10}$$

$$d\boldsymbol{r}_f = e_f s\Lambda\boldsymbol{i} + e_f c\Lambda\boldsymbol{j} + e_f\Gamma\boldsymbol{k} \tag{11}$$

Performing the indicated cross products in Eq. (7) and simplifying yields the moments applied to the airframe by sloshing fuel as

$$\begin{aligned} dL/m_f &= -l_0\Gamma s\Lambda\left(\omega^2 e_f + 2\zeta\omega\dot{e}_f\right) + ge_f c\Lambda c\Theta_0 \\ dM/m_f &= -l_0\Gamma c\Lambda\left(\omega^2 e_f + 2\zeta\omega\dot{e}_f\right) - ge_f\Gamma s\Theta_0 - ge_f s\Lambda c\Theta_0 \\ dN/m_f &= l_0\left(\omega^2 e_f + 2\zeta\omega\dot{e}_f\right) + ge_f c\Lambda s\Theta_0 \end{aligned} \tag{12}$$

Modal Coupling

Coupled fuel slosh-to-aircraft small-perturbation equations are useful for control system synthesis and also as an aid to understanding complete nonlinear motions with fuel slosh coupling. The coupled small-perturbation equations are developed by adding fuel motion degrees of freedom to the normal small-perturbation aircraft equations, much as one adds structural flexibility normal modes to the rigid equations of motion for aeroelastic analysis.

Only the fuel-coupled lateral or asymmetric small-perturbation equations of motion are developed here. This is because fuel slosh coupling through internal forces on the tanks enters the longitudinal equations through variations in the X-axis force. This gives only weak coupling into the longitudinal short-period mode, usually well approximated by neglecting the longitudinal degree of freedom. For the swept wing tank case, there is a gravity pitching moment coupling into the long-period or phugoid motion. This gravity moment coupling is demonstrated in the numerical example to follow.

The coupled lateral small-perturbation equations are, in state matrix form,

$$A\dot{x} = Bx + Cu \qquad \text{or} \qquad \dot{x} = A^{-1}Bx + A^{-1}Cu \tag{13}$$

The extended state vector x is $\{\beta \;\; \phi \;\; p \;\; \psi \;\; r \;\; e_f \;\; \dot{e}_f\}$, where the last two elements are fuel displacement and displacement rate, respectively. The control vector u is $\{\delta_a \;\; \delta_r\}$. The A matrix is a 7×7 unit matrix with the following additional nonzero terms:

$$A(3,5) = -I_{xz}/I_x \qquad A(7,3) = -l_0\Gamma s\Lambda$$

$$A(5,3) = -I_{xz}/I_z \qquad A(7,5) = l_0 \qquad A(7,1) = V_{t0}c\Lambda$$

The upper left 5×5 portion of the B matrix and the C matrix are the standard aircraft small-perturbation forms. Unprimed[6] derivatives are used since the prime matrix inversion is part of Eq. (13). Using standard definitions for the derivatives,[6] nonzero elements of the upper 5×5 matrix portion of B and the C matrix are

$$B(1,1) = Y_v \qquad C(1,1) = Y^*_{\delta a}$$

$$B(1,2) = gc\Theta_0/V_{t0} \qquad C(1,2) = Y^*_{\delta r}$$

$$B(1,3) = W_0/V_{t0} \qquad C(3,1) = L_{\delta a}$$

$$B(1,5) = Y^*_r - U_0/V_{t0} \qquad C(3,2) = L_{\delta r}$$

$$B(2,3) = 1 \qquad C(5,1) = N_{\delta a}$$

$$B(2,5) = \tan\Theta_0 \qquad C(5,2) = N_{\delta r}$$

$$B(3,1) = L_\beta \qquad B(3,3) = L_p$$

$$B(3,5) = L_r \qquad B(4,5) = 1/c\Theta_0$$

$$B(5,1) = N_\beta \qquad B(5,3) = N_p \qquad B(5,5) = N_r$$

The remaining nonzero terms of the B matrix are contributed by fuel dynamics:

$$B(1,6) = (m_f/mV_{t0})\omega^2 c\Lambda \qquad B(6,7) = 1$$

$$B(1,7) = (m_f/mV_{t0})2\zeta\omega c\Lambda \qquad B(7,2) = gc\Theta_0 c\Lambda$$

$$B(3,6) = (m_f/I_x)\big(-l_0\Gamma\omega^2 s\Lambda + gc\Lambda c\Theta_0\big) \qquad B(7,3) = W_0 c\Lambda$$

$$B(3,7) = -(m_f/I_x)l_0\Gamma 2\zeta\omega s\Lambda \qquad B(7,4) = gs\Theta_0 c\Lambda$$

$$B(5,6) = (m_f/I_z)\big(l_0\omega^2 + gc\Lambda s\Theta_0\big) \qquad B(7,5) = -U_0 c\Lambda$$

$$B(5,7) = (m_f/I_z)l_0 2\zeta\omega \qquad B(7,6) = -\omega^2 \qquad B(7,7) = -2\zeta\omega$$

The coupled small-perturbation state equations (13) are augmented by additional state equations for control system compensation and servodynamics.

Reduced Degrees of Freedom

Each sloshing fuel mass produces an additional degree of freedom for the coupled system and an additional mode of motion. However, for an important case, two slosh masses act to produce only one additional degree of freedom. This is the case for left and right partially filled tanks with the same fuel loads. This happens because in the A and B matrix elements of Eq. (13) when the sweep angle Λ appears as the sine function it is always multiplied by the dihedral angle Γ; when Λ appears alone it is always as a cosine function. Since Λ and Γ change signs together from left- to right-wing panels, A and B matrix elements are identical for left- and right-wing tanks with the same fuel loads.

Thus for the case of left and right tanks with the same partial fuel load, the correct modes and roots are found by using parameters for one side only, but with twice the fuel mass for that side. This results in only two fuel slosh states, corresponding to fuel displacement e_f and fuel displacement rate $\dot{e}_f$. The effect of allocating four states in this case, two for each tank, is discussed in the Locus of Roots with Fuel Mass as Gain subsection.

Slosh Frequency and Damping

Lamb[3] gives the normal modal frequencies of the free oscillation of a fluid bounded by vertical walls as

$$\omega = (n\pi/a)\sqrt{gd}, \qquad d/a \ll 1 \tag{14}$$

where n is 1 for the fundamental or lowest frequency. This simple expression holds for small values of d/a, typical of wing fuel tanks. Equation (14) overestimates natural frequency for deep tanks, such as found in airplane fuselages. In that case, only the top layer of fluid sloshes, and frequency is a function of length only. This limiting case is[2]

$$\omega = \sqrt{\pi g/a}, \qquad d/a \gg 1 \tag{15}$$

The inherent damping of fuel sloshing in an unobstructed tank is very low. However, internal wing structure such as ribs, spars, and skin stiffeners can be counted on to raise slosh damping considerably. Slosh damping can be measured in model tests for cases in which it is important to have a precise value. A slosh damping ratio of 0.5 is taken quite arbitrarily as the baseline for the example cases to follow.

Slosh Limits and Wing Dihedral

Fuel slosh dynamics, especially for the long, thin tanks characteristic of wing tanks, are profoundly affected by hard limits in the fuel's travel. Two important limiting effects are as follows.

1) For motions with stable complex roots without fuel slosh and unstable complex roots with fuel slosh, the actual motion will be a limit cycle.

2) For swept wings, fuel is free to slosh only when the airplane is flying over a limited pitch attitude range.

The second effect is explained by Figs. 2 and 3. Figure 2 shows the effective linear range for the fuel centroidal shift from tank center as a function of a fraction of tank fill. The dashed line is for the maximum possible shift, occurring when the tank is on its side, rolled 90 deg. The solid line, not far below the dotted line, is for the free surface just in contact with the tank roof. With tanks that are almost full, only small centroidal shifts are possible before a hard limit is reached. A shift of a little over 10% of tank length is possible for three-quarter full tanks.

Figure 3 is derived from the steady-state solution of the fuel motion equation (5), at zero bank angle:

$$\omega^2 e_f/g = -s\Lambda s\Theta + \Gamma c\Theta \tag{16}$$

Figure 3 is plotted for the example case wing sweep of 37 deg. Wing fuel is within its linear slosh range over only a small pitch attitude interval.

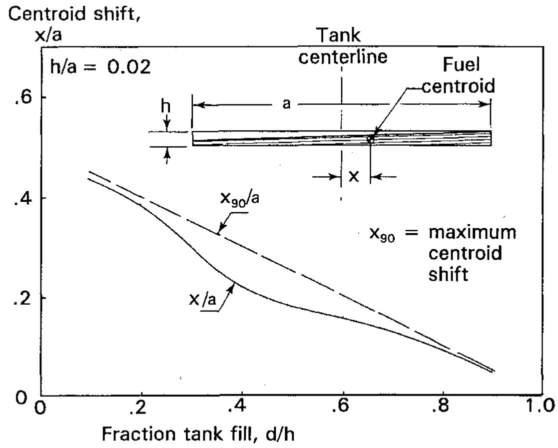

Fig. 2 Effective linear range for fuel centroid shift from centerline (free surface just meeting the tank roof).

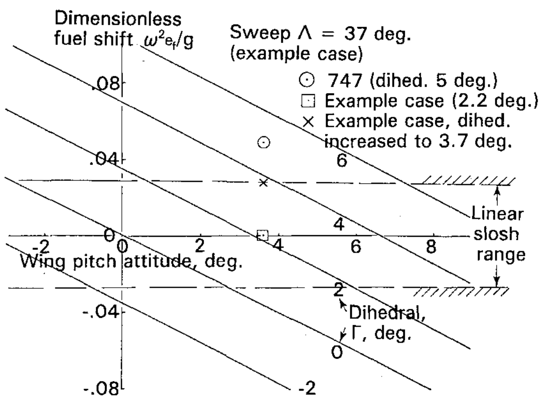

Fig. 3 Variation of steady-state dimensionless fuel shift with wing dihedral and wing pitch attitude (airplane plus incidence).

Example Skewed Wing Tank Case

Example root and transient time history calculations are made for lateral fuel slosh using data for the Boeing 747-100 airplane. Numerical values such as weights, dimensions, and stability derivatives are furnished by Heffley and Jewell.[7] Wing tank dimensions and loads were kindly furnished by the Boeing Company, Commercial Airplane Division. The airplane is at 564,032 lb, at 165 kn, in landing approach. A low airspeed provides the maximum possibility for fuel slosh forces to be significant relative to aerodynamic forces. A simplified yaw damper is operating, with rudder to yawing velocity gain of −3.98 and washout time constant of 1/0.368 s. Second-order rudder servodynamics are assumed.

At the pitch attitude corresponding to this approach condition, the 747's actual wing dihedral of 5 deg puts the wing fuel outside of the slosh range, as shown by the circle symbol in Fig. 3. Accordingly, a dihedral angle of 2.2 deg is arbitrarily used in the calculations, to put the wing fuel in the center of its slosh range.

Locus of Roots with Fuel Mass as Gain

Figure 4 is a locus of roots for the inboard main wing tanks, with fuel mass as the gain parameter. There are 10 poles, corresponding to no fuel, and 9 finite zeros, for infinite fuel mass. The 10 poles are identified as follows: 5 for the airframe, 2 for fuel, 2 for a second-order rudder servo, and 1 for a high-pass filter (washout) compensation. The square plot symbols correspond to three-quarter full fuel. The root locus branches originating at rudder servo poles are outside the range of the plot. In what follows, root branches are identified with the poles from which they originate; e.g., D for Dutch roll, S for spiral, and W for yaw damper washout loop refer to modes as modified by the closure.

The Dutch roll mode frequency is increased, and the slosh mode frequency is decreased as fuel is added. A small unstable real root, similar to an unstable spiral mode, appears as fuel is added. The root value is 0.053 at three-quarter full fuel, corresponding to a time to double amplitude of 13 s. This unstable mode could appear to the pilot as wing heaviness. The root branch originating at the airplane's spiral mode actually increases negatively (more stable) as fuel is added.

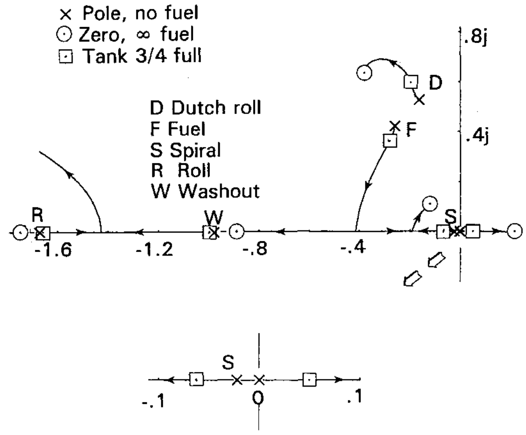

Fig. 4 Locus of lateral roots with inboard main wing fuel mass as gain.

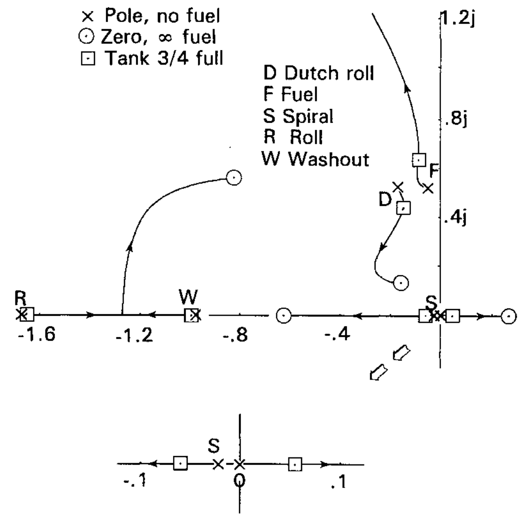

Fig. 5 Locus of lateral roots with inboard main wing fuel mass as gain, slosh damping ratio reduced to 0.1.

When four states are allocated to sloshing fuel by duplicating the equations to represent a second tank, the slosh pole is repeated. The repeated pole is overlaid by a zero, and the remainder of the locus of roots is identical to the two-state case. The conclusion must be that the two-state slosh model for identical left- and right-wing tanks is adequate for control system synthesis by any linearized method.

A hypothetical 70-ft-long wing tank is formed by combining the main inboard and outboard tanks. For this case, the unstable real root is increased from 0.053 to 0.107, at the same fuel weight.

Effect of Low Slosh Damping

Figure 5 shows the effect of reducing the fuel slosh damping ratio from the assumed value of 0.5 to 0.1. The root locus looks different from Fig. 4, but results are similar for three-quarter full tanks (squares). Damping of the modes of motion is unchanged, as is the unstable real root.

Conventional Root Locus

Sloshing fuel as an additional degree of freedom should be considered for inclusion into airframe models for control system synthesis, even as structural flexibility modes are normally considered for inclusion. However, whether or not to complicate airframe models by fuel slosh or aeroelastic modes must be determined for each design.

As an example of how an added skewed-tank fuel slosh mode might affect control system synthesis, conventional yaw damper

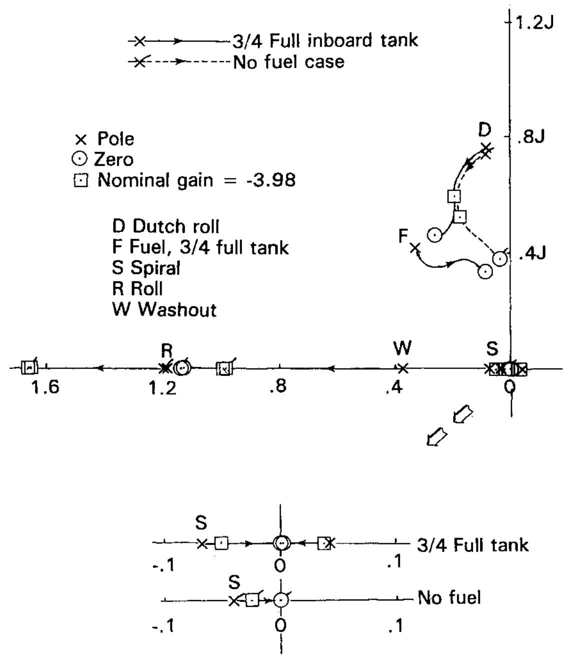

Fig. 6 Comparison of conventional yaw damper root loci for airframe models with and without fuel slosh. Dutch roll branch mainly affected by slosh; aperiodic divergence added.

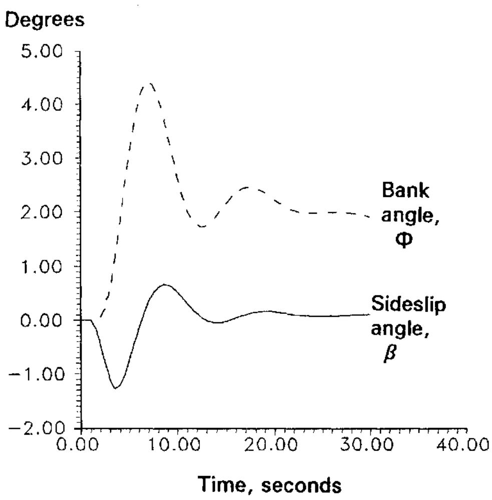

Fig. 7 Bank and sideslip responses to 5-deg right-rudder pulse, inboard main wing tank three-quarter full.

root loci are presented for the example skewed wing tank case, with and without a slosh fuel mode. The skewed inboard main tanks are taken as three-quarter full. The loci are compared in Fig. 6, with the no-slosh case shown dashed and with ticked symbols. In each case, there are three more poles than finite zeros, putting asymptotes along the negative real axis and at ±60 deg (off scale).

There are three fewer locus branches for the case without slosh instead of the two fewer that might be expected because of a dipole at the origin (not shown), corresponding to the heading perturbation ψ degree of freedom. Perturbation angle ψ couples with sloshing fuel [matrix term $B(7,4)$], but provides a zero mode for the no-fuel case, as is normal.

The root locus branch originating at the Dutch roll pole is somewhat modified by sloshing fuel, and a small unstable real root appears. In this case, adding the skewed wing tank mode to the yaw damper synthesis would probably lead to choice of a slightly higher gain than for the case with the slosh mode neglected.

Transient Response

Transient responses to a 3-s right rudder pulse of 5 deg, starting at 1 s, are shown in Fig. 7, for the three-quarter full wing tank

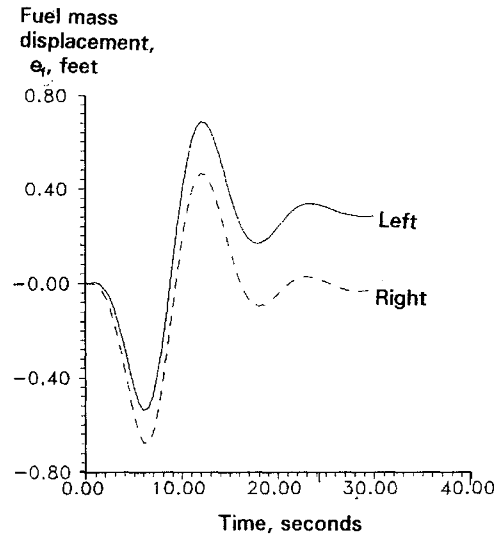

Fig. 8 Fuel mass responses to 5-deg right-rudder pulse, inboard main wing tank three-quarter full.

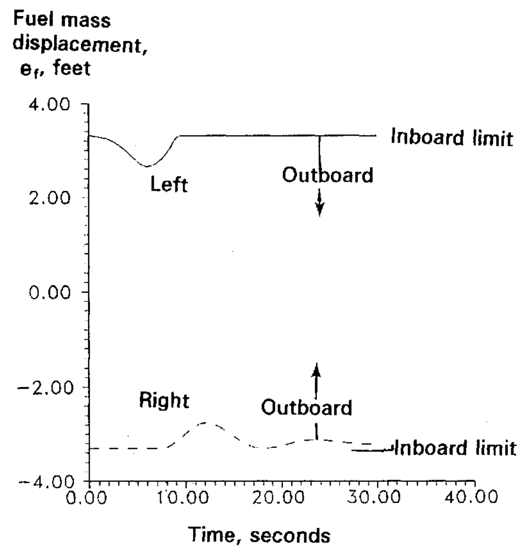

Fig. 9 Fuel mass responses to 5-deg right-rudder pulse, inboard main wing tank three-quarter full. Dihedral is increased to 3.7 deg, putting fuel at the edge of the linear slosh range.

case corresponding to the squares of Fig. 4. Full 12-state nonlinear equations of airframe motion are used, together with the nonlinear fuel motion equation (5), applied to each tank independently, adding four more states. The yaw damper with washout and second-order rudder servo dynamics are also represented.

Sideslip and bank angle oscillate at the frequency predicted in the linearized analysis for the modified Dutch roll. Sideslip returns to zero, but the airplane remains in a 2-deg right-banked turn after the oscillations die away. The right-banked attitude following rudder pulse occurs as well for the case of fixed fuel.

As shown in Fig. 8, both fuel masses move synchronously at the predicted frequency for the modified Dutch roll, agreeing with the linearized finding of just one mode for the two equal masses. However, as a result of the stabilized right bank at the end of the run, there is a bias of 0.13 ft in total fuel mass to the right. The bias is the average fuel mass displacement. In addition to the bias, there is a net forward fuel displacement (left fuel inboard), caused by a slight nose-down pitch change during most of the run. The record is not long enough to detect the predicted slow aperiodic divergence.

Transient Response, Increased Dihedral

Figure 9 shows fuel motion for the interesting case of assumed wing dihedral equal to the × symbol of Fig. 3. This is on one boundary of the linear slosh range for the example airplane at a pitch attitude of 3.7 deg. At the dihedral angle of 3.7 deg, fuel in the main inboard wing tanks is just at the inboard slosh limit with the airplane in undisturbed flight. Dutch roll disturbances cause fuel to move outboard, but not inboard, as shown in Fig. 9. Coupling is greatly reduced.

Example Centerline Tank Case

Perspective on skewed tank slosh coupling is given by comparison with a centerline, unskewed tank case. Figure 10 gives the root locus for the centerline case, representing the Boeing 747's center fuel tank. Coupling with the airplane's Dutch roll mode is minimal.

Effect of Tank Fore-and-Aft Location

Figure 11 shows the effect of tank fore-and-aft location on the zero of the Dutch roll dipole of Fig. 10. This is an example of the classic dipole effect discussed by McRuer and Johnston.[8] In the upper-half of the s plane, zero locations below the pole produce stable closures as gain is increased; i.e., the locus from pole to zero, if drawn, would make a semicircle to the left, away from the imaginary axis. The opposite is true for zero locations above the pole.

At the same time, the close proximity of the pole and zero signifies a small contribution of that mode to the transient response, or little slosh coupling to the Dutch roll.

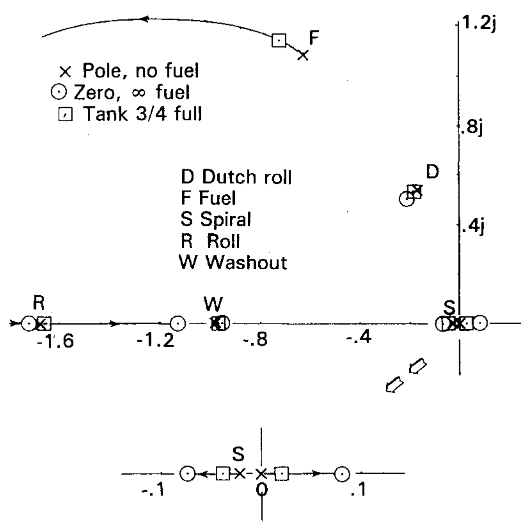

Fig. 10 Locus of lateral roots with centerline tank (unskewed) fuel mass as gain.

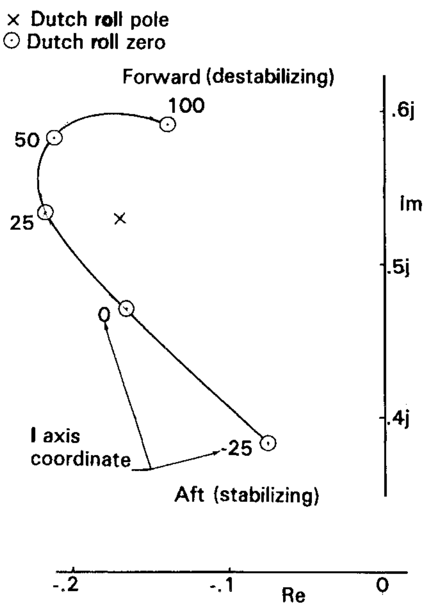

Fig. 11 Effect of tank fore-and-aft location on Dutch roll mode dipole zero.

Zero Side Force Case

The flight dynamics of aircraft with very low side force derivatives such as Y_v and $Y_{\delta r}$ are of interest because values of this sort are associated with aircraft designed for low radar returns, without vertical tails and with blended fuselages (Northrop Grumman B-2 and Lockheed Martin/Boeing Tier 3). Low side force derivatives remove one source of excitation for sloshing fuel.

This is confirmed by root locus and transient analysis, not presented for the sake of brevity, of the original skewed tank example configuration in which all side force derivatives are arbitrarily set at zero. Fuel excursions following rudder pulses are about half of those shown in Fig. 8.

Conclusions

The following conclusions are drawn from this study.

1) Fuel in long, narrow wing tanks has its primary, low-frequency slosh mode along the tank long axis, regardless of wing sweep and direction of forcing input.

2) A locus of roots of the lateral characteristic equation with sloshing fuel mass as a gain parameter shows coupling of the fuel slosh and Dutch roll modes, and a long time constant unstable spiral-like mode of motion.

3) The unstable spiral-like mode, appearing with conventional yaw damper on or off, is larger for longer wing fuel tanks, which have lower slosh mode natural frequencies.

4) Only one additional asymmetric mode, the fuel mode, is created in linearized analysis for two independent equal slosh fuel masses in the left and right wings. This is confirmed in nonlinear transient responses with tanks represented independently. For either rudder or aileron pulses, left and right masses move synchronously, in the same direction.

5) Fuel–Dutch roll coupling is relatively small for the subsonic swept wing configuration examined, with a wing tank slosh mass to gross weight ratio of 0.11. The Dutch roll mode frequency is increased; the slosh mode frequency is decreased. Even smaller effects are seen for a centerline (unskewed) tank with slosh mass to gross weight ratio of 0.15.

6) Fuel slosh magnitude is approximately halved for the case in which all side force derivatives are set to zero. Small or zero side force coefficients are found in configurations designed for low radar returns.

7) The dihedral of swept wings has a powerful limiting effect on fuel slosh coupling with the modes of motion. For any dihedral and wing sweep, there is only a limited pitch attitude range where significant slosh can occur.

In conclusion, this analysis demonstrates a method for incorporating fuel slosh into both linearized and full nonlinear equations of airplane motion, for the case of tanks skewed to the direction of flight. One fuel slosh mode is added in the linearized case, typically used for control system synthesis. Modes are added for each partially filled tank for the nonlinear case, typically used for control system analysis and simulation. The situation is analogous to structural mode coupling into the equations of rigid-body motion. Whether available slosh or aeroelastic models are used or not must be decided for each case.

References

[1]Abzug, M. J., "Effects of Fuel Slosh on Stability and Control," Douglas Aircraft Co., Rept. No. ES 29551, Long Beach, CA, July 1959.

[2]Graham, E. W., "The Forces Produced by Fuel Oscillation in a Rectangular Tank," Douglas Aircraft Co., Rept. No. SM 13748, Long Beach, CA, April 1950.

[3]Lamb, H., *Hydrodynamics*, 6th ed., Dover, New York, 1945, pp. 282, 284.

[4]Schy, A. A., "A Theoretical Analysis of the Effects of Fuel Motion on Airplane Dynamics," NACA Rept. 1080, 1952.

[5]Luskin, H., and Lapin, E., "An Analytical Approach to the Fuel Sloshing and Buffeting Problems of Aircraft," *Journal of the Aeronautical Sciences*, Vol. 8, No. 4, 1952, pp. 217–228.

[6]McRuer, D., Ashkenas, I., and Graham, D., *Aircraft Dynamics and Automatic Control*, Princeton Univ. Press, Princeton, NJ, 1973, p. 295.

[7]Heffley, R. K., and Jewell, W. F., "Aircraft Handling Qualities Data," NASA CR-2144, Dec. 1972.

[8]McRuer, D. T., and Johnston, D. E., "Flight Control Systems Properties and Problems, Vol. I," NASA CR-2500, Feb. 1975.